A Table of Elliptic Integrals of the Third Kind*

By B. C. Carlson

Abstract. As many as 72 elliptic integrals of the third kind in previous tables are unified by evaluation in terms of R-functions instead of Legendre's integrals. The present table includes only integrals having integrands with real singular points. In addition to 31 integrals of the third kind, most of them unavailable elsewhere, 10 integrals of the first and second kinds from an earlier table are listed again in new notation. In contrast to conventional tables, the interval of integration is not required to begin or end at a singular point of the integrand. Fortran codes for the standard R-functions R_C and R_J, revised to include their Cauchy principal values, are listed in a Supplement.

1. Introduction. A table [4] of elliptic integrals of the first and second kinds is extended in this paper to integrals of the third kind. We choose a standard form,

$$(1.1) \qquad [p] = [p_1, \ldots, p_n] = \int_y^x \prod_{i=1}^n (a_i + b_i t)^{p_i/2} \, dt,$$

where $p_1, \ldots, p_n$ are integers and the integrand is real. The integral is assumed to be well defined; in particular, if p_i is odd, $a_i + b_i t$ must be positive in the open interval of integration. Many integrals like

$$\int (1 + n \sin^2 \phi)^{-1} (1 - k^2 \sin^2 \phi)^{-1/2} \, d\phi$$

and

$$\int (1 + nz^2)^{-1} [(1 - z^2)(1 - k^2 z^2)]^{-1/2} \, dz$$

can be put in the form (1.1) by letting $t = \sin^2 \phi$ or $t = z^2$.

If we assume the b's are nonzero and no two of the quantities $a_i + b_i t$ are proportional, then $[p_1, \ldots, p_n]$ is an elliptic integral if the number of odd p's is exactly three (the "cubic case") or four (the "quartic case"). It is elliptic of third kind if at least one of the p's is even and negative or if $p_1 + \cdots + p_n = -2, 0, 2, 4, \ldots$, the latter condition being impossible in the cubic case. Otherwise, it is first or second kind, and the only integrals of first kind are $[-1, -1, -1]$ and $[-1, -1, -1, -1]$.

The conditions that the integral be of third kind are deduced by using [2,(8.1-2)] to express $[p]$ as an R-function with "b-parameters" $-p_1/2, \ldots, -p_n/2, 2 + \sum p_i/2$. The integral is of third kind if at least one of these parameters is a positive integer (see [2, §8.5, §9.2]).

We shall replace Legendre's integrals of the first and third kinds by

$$(1.2) \qquad R_F(x,y,z) = \frac{1}{2} \int_0^\infty [(t+x)(t+y)(t+z)]^{-1/2} / \, dt,$$

$$(1.3) \qquad R_J(x,y,z,\rho) = \frac{3}{2} \int_0^\infty [(t+x)(t+y)(t+z)]^{-1/2}(t+\rho)^{-1} \, dt.$$

Besides being symmetric in x, y, z, these R-functions are homogeneous:

$$(1.4) \qquad \begin{aligned} R_F(\lambda x, \lambda y, \lambda z) &= \lambda^{-1/2} R_F(x,y,z), \\ R_J(\lambda x, \lambda y, \lambda z, \lambda \rho) &= \lambda^{-3/2} R_J(x,y,z,\rho), \end{aligned}$$

and they are normalized so that

$$(1.5) \qquad R_F(x,x,x) = x^{-1/2}, \qquad R_J(x,x,x,x) = x^{-3/2}.$$

Two special cases are denoted by

$$(1.6) \qquad R_C(x,y) = R_F(x,y,y), \qquad R_D(x,y,z) = R_J(x,y,z,z).$$

The function R_D replaces Legendre's integral of the second kind, and R_C is an elementary function** embracing the logarithmic, inverse circular, and inverse hyperbolic functions [3, (4.9)–(4.13)]. Use of R_C allows unification of formulas for circular and hyperbolic cases of integrals of the third kind.

Fortran codes [5] for R_C and R_J are listed in the Supplements section of this issue, and codes for R_F and R_D are given in a Supplement to [4]. All four can be found also in most of the major software libraries, but the codes for R_C and R_J in the Supplement have recently been modified to compute the Cauchy principal value of the integral when the last variable is negative. As shown in Sections 5 and 6, Cauchy principal values are sometimes needed when using the formulas of this paper.

As explained in [4], the present table differs from customary integral tables [1], [7]***, [8] in two respects: We do not require the interval of integration to begin or end at a branch point of the integrand, and we do not separate special cases according to the positions of the branch points relative to the interval of integration and to one another. The unification of these special cases is made possible by application of the addition theorem and by the use of R-functions instead of Legendre's integrals. The present table includes only integrands with real branch points; conjugate complex branch points resulting from an irreducible polynomial $a_i + b_i t + c_i t^2$ are deferred to a later paper.

The table in Section 2 consists of quartic cases, since many cubic cases are included in these by taking $a_i = 1$ and $b_i = 0$ for various values of i. To select integrals that are relatively simple, we arbitrarily require $\sum |p_i| \leq 8$ and $\sum p_i \leq 0$. Apart from permutation of subscripts in (1.1), there are just 37 quartic cases satisfying these criteria. We omit nine with $p_5 = \pm 2$, $p_6 = \pm 2$ since they can

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**Although not in connection with elliptic integrals, the use of a function equivalent to the reciprocal of R_C was proposed in 1897 by Fubini, while still a student at Pisa, in his first published paper [6]. I am obliged to Professor Luigi Gatteschi for this reference.

***Tables 3 and 4 in [7] use Legendre's Π as defined by [1, (110.04)] and are inconsistent with [7, 8.111(4)], which differs in the sign of the parameter. The same is true of the 1965 edition.

easily be reduced to the others by (2.51)–(2.53), but we add three with $\sum p_i >$ 0: $[1, 1, 1, -1]$, $[1, 1, 1, 1]$, and $[1, 1, 1, 1, -2]$. Thus we list 31 quartic cases, most of which are not available in previous tables. The formulas for $[1, -1, -1, -1]$, $[1, 1, -1, -3]$, $[1, 1, -1, -1]$, and $[1, 1, 1, -1]$ respectively unify 32, 72, 48, and 32 special cases in Byrd and Friedman's table [1].

All cubic cases of third kind with $\sum |p_i| \leq 7$ are contained in the quartic cases of Section 2 with three exceptions: $[1, 1, 1, -2]$, $[3, 1, -1, -2]$, and $[3, 1, 1, -2]$. Cubic cases of all three kinds will be discussed in a later paper.

Because some of the formulas are cumbersome when written out in the style of [4], we have introduced abbreviations to save space. For uniformity of style, the quartic cases of first and second kinds listed in [4] are listed again in Section 3 in the new notation. Derivation of formulas by recurrence relations is discussed in Section 4. The fundamental formula for $[1, -1, -1, -1, -2]$ is proved in Section 5, and Cauchy principal values are discussed in Section 6. All integral formulas have been checked by numerical integration; some details of the checks are given in the Supplement.

2. Table of Quartic Cases.

We assume $x > y$ and $a_i + b_i t > 0$, $y < t < x$, for $i = 1, \ldots, 4$. Assumptions about $a_5 + b_5 t$ will be stated where necessary. We define

$$(2.1) \qquad d_{ij} = a_i b_j - a_j b_i, \qquad r_{ij} = \frac{d_{ij}}{b_i b_j} = \frac{a_i}{b_i} - \frac{a_j}{b_j};$$

$$(2.2) \qquad X_i = (a_i + b_i x)^{1/2}, \qquad Y_i = (a_i + b_i y)^{1/2};$$

$$(2.3) \qquad U_{ij} = (X_i X_j Y_k Y_m + Y_i Y_j X_k X_m)/(x - y),$$

where i, j, k, m is any permutation of $1, 2, 3, 4$;

$$(2.4) \qquad W^2 = U_{12}^2 - d_{13} d_{14} d_{25}/d_{15};$$

$$(2.5) \qquad Q^2 = (X_5 Y_5 W / X_1 Y_1)^2, \qquad P^2 = Q^2 + d_{25} d_{35} d_{45}/d_{15};$$

$$(2.6) \qquad A(p_1, \ldots, p_n) = X_1^{p_1} \cdots X_n^{p_n} - Y_1^{p_1} \cdots Y_n^{p_n}.$$

These definitions imply, if P is chosen positive,

$$(2.7) \qquad U_{ij}^2 - U_{ik}^2 = d_{im} d_{jk},$$

$$(2.8) \qquad P = (X_1^{-1} X_2 X_3 X_4 Y_5^2 + Y_1^{-1} Y_2 Y_3 Y_4 X_5^2)/(x - y),$$

and hence, with the help of [4, (5.22)],

$$(2.9) \qquad \begin{aligned} U_{13}^2 = U_{12}^2 - d_{14} d_{23}, \qquad U_{14}^2 = U_{12}^2 - d_{13} d_{24}, \\ W^2 = U_{1i}^2 - d_{1j} d_{1k} d_{i5}/d_{15}, \end{aligned}$$

where i, j, k is any permutation of $2, 3, 4$. When $a_5 = 1$ and $b_5 = 0$, the quantities W, P, Q become W_1, P_1, Q_1:

$$(2.10) \qquad \begin{aligned} W_1^2 = U_{12}^2 - b_2 d_{13} d_{14}/b_1, \\ Q_1^2 = (W_1/X_1 Y_1)^2, \qquad P_1^2 = Q_1^2 + b_2 b_3 b_4/b_1. \end{aligned}$$

If one limit of integration is infinite, (2.3) simplifies to

$$(2.11) \qquad \begin{aligned} U_{ij} = (b_i b_j)^{1/2} Y_k Y_m + Y_i Y_j (b_k b_m)^{1/2}, \qquad x = +\infty, \\ U_{ij} = X_i X_j (b_k b_m)^{1/2} + (b_i b_j)^{1/2} X_k X_m, \qquad y = -\infty, \end{aligned}$$

all square roots being nonnegative, while

(2.12)
$$Q^2 = (b_5/b_1)(Y_5W/Y_1)^2, \qquad x = +\infty,$$
$$Q^2 = (b_5/b_1)(X_5W/X_1)^2, \qquad y = -\infty.$$

All quartic cases will be expressed in terms of the quantities

(2.13)
$$I_1 = 2R_F(U_{12}^2, U_{13}^2, U_{14}^2),$$

(2.14)
$$I_2 = \frac{2}{3}d_{12}d_{13}R_D(U_{12}^2, U_{13}^2, U_{14}^2) + \frac{2X_1Y_1}{X_4Y_4U_{14}},$$

(2.15)
$$I_3 = \frac{2d_{12}d_{13}d_{14}}{3d_{15}}R_J(U_{12}^2, U_{13}^2, U_{14}^2, W^2) + 2R_C(P^2, Q^2),$$

(2.16)
$$I_3' = \frac{-2d_{12}d_{13}d_{14}}{3b_1}R_J(U_{12}^2, U_{13}^2, U_{14}^2, W_1^2) + 2R_C(P_1^2, Q_1^2).$$

It will be seen from the tables that

(2.17)
$$I_1 = [-1, -1, -1, -1], \qquad I_2 = [1, -1, -1, -3],$$
$$I_3 = [1, -1, -1, -1, -2], \qquad I_3' = [1, -1, -1, -1].$$

Thus, I_3 reduces to I_3' if $a_5 = 1$ and $b_5 = 0$, and I_3 reduces to I_2 if $a_5 = a_4$ and $b_5 = b_4$.

If one limit of integration is a branch point of the integrand, then X_i or Y_i is 0 for some value of $i \leq 4$, and one of the two terms on the right-hand side of (2.3) vanishes. If $X_1Y_1 = 0$ then P, Q, P_1, and Q_1 are infinite, and the R_C-functions in (2.15) and (2.16) vanish by (1.6) and (1.4). If both limits of integration are branch points, the elliptic integral is called complete, and $U_{12}U_{13}U_{14} = 0$. It is not assumed that $b_i \neq 0$ nor that $d_{ij} \neq 0$, unless one of these quantities occurs in a denominator. The relation $d_{ij} = 0$ is equivalent to proportionality of $a_i + b_i t$ and $a_j + b_j t$.

We are now ready to list 31 cases of

(2.18)
$$[p_1, \ldots, p_5] = \int_y^x (a_1 + b_1 t)^{p_1/2} \cdots (a_5 + b_5 t)^{p_5/2} \, dt,$$

15 with $p_5 = 0, 2$, or 4 and 16 with $p_5 = -2$ or -4. The first 10 have $p_5 = 0$ and involve I_3' but not I_3.

(2.19)
$$[1, -1, -1, -1] = I_3'.$$

(2.20)
$$[1, 1, -1, -3] = (b_2I_3' + d_{24}I_2)/b_4.$$

(2.21)
$$[3, -1, -1, -3] = (b_1I_3' + d_{14}I_2)/b_4.$$

(2.22)
$$[1, 1 - 1, -1] = [b_2(r_{13} + r_{24})I_3' + d_{24}r_{34}I_2 - d_{12}r_{13}I_1]/2$$
$$+ A(1, 1, 1, -1)/b_3.$$

(2.23)
$$[3, -1, -1, -1] = (b_1/2)\left[\sum_{i=2}^{4} r_{1i}I_3' + b_4r_{24}r_{34}I_2 \right.$$
$$\left. - b_1r_{12}r_{13}I_1 + 2A(1, 1, 1, -1)/b_2b_3 \right].$$

$$(2.24) \quad [1,1,1,-3] = (b_2 b_3 / 2b_4) \left[\sum_{i=1}^{3} r_{i4} I_3' + 3b_4 r_{24} r_{34} I_2 - b_1 r_{12} r_{13} I_1 \right]$$
$$+ A(1,1,1,-1)/b_4.$$

$$(2.25) \quad [3,1,-1,-3] = (b_1 b_2 / 2b_4)[(r_{13} + 2r_{14} + r_{24})I_3'$$
$$+ b_4 r_{24}(2r_{14} + r_{34})I_2 - b_1 r_{12} r_{13} I_1]$$
$$+ (b_1 / b_3 b_4) A(1,1,1,-1).$$

$$(2.26) \quad [1,1,1,-5] = (b_2 b_3 / b_4^2) I_3' + (d_{24} d_{34} / 3b_4^3) \sum_{i=1}^{3} r_{i4}^{-1} I_2$$
$$- (d_{12} d_{13} / 3b_4 d_{14}) I_1 - (2/3b_4) A(1,1,1,-3).$$

$$(2.27) \quad [1,1,1,-1] = (b_2 b_3 / 8)(r_{14}^2 - r_{12}^2 - r_{13}^2 + 2r_{24} r_{34}) I_3'$$
$$+ \left(\sum_{i=1}^{3} r_{i4}/8 \right) [b_3 d_{24} r_{34} I_2 + 2A(1,1,1,-1)]$$
$$+ (b_3 d_{12} r_{13}/8)(r_{12} - r_{34}) I_1 + A(1,1,1,1)/2b_4.$$

$$(2.28) \quad [1,1,1,1] = (b_2 b_3 b_4 / 16)(r_{12} - r_{34})(r_{14}^2 - r_{23}^2) I_3'$$
$$- [(r_{12} - r_{34})^2 + 2(r_{14}^2 + r_{23}^2)]$$
$$\cdot [d_{24} d_{34} I_2 + 2b_4 A(1,1,1-1)]/48$$
$$+ (b_3 b_4 d_{12} r_{13}/48)(4r_{24} r_{34} - 3r_{14}^2 + 3r_{23}^2) I_1$$
$$- \left(\sum_{i=2}^{4} r_{1i}/12 \right) A(1,1,1,1) + A(3,1,1,1)/3b_1.$$

The next five integrals, with $p_5 = 2$ or 4, involve I_3' but not I_3. No restriction is placed on a_5 or b_5.

$$(2.29) \quad [-1,-1,-1,-1,2] = (b_5 I_3' - d_{15} I_1)/b_1.$$

$$(2.30) \quad [1,-1,-1,-1,2] = (b_5/2)[(r_{12} - r_{35} - r_{45}) I_3' + b_4 r_{24} r_{34} I_2$$
$$- b_1 r_{12} r_{13} I_1 + (2/b_2 b_3) A(1,1,1,-1)].$$

$$(2.31) \quad [1,-1,-1,-3,2] = (b_5 I_3' - d_{45} I_2)/b_4.$$

$$(2.32) \quad [1,1,-1,-3,2] = (b_2 b_5 / 2b_4)[(r_{13} + r_{24} - 2r_{45}) I_3'$$
$$+ b_4 r_{24}(r_{34} - 2r_{45}) I_2 - b_1 r_{12} r_{13} I_1]$$
$$+ (b_5 / b_3 b_4) A(1,1,1,-1).$$

$$[-1,-1,-1,-1,4]$$
$$(2.33) \quad = (b_5^2 / 2b_1) \left[- \sum_{i=1}^{4} r_{i5} I_3' + b_4 r_{24} r_{34} I_2 + b_1 (2r_{15}^2 - r_{12} r_{13}) I_1 \right.$$
$$\left. + (2/b_2 b_3) A(1,1,1,-1) \right].$$

The next 11 integrals have $p_5 = -2$, and we assume $a_5 + b_5 t$ either is positive on the closed interval of integration or changes sign in the open interval of integration. In the latter case the integral is interpreted as a Cauchy principal value; see Section 6. All 11 integrals involve I_3, and the six with $\sum p_i \geq -2$ involve I_3'/b_5 also.

(2.34)
$$[1, -1, -1, -1, -2] = I_3.$$

(2.35)
$$[-1, -1, -1, -1, -2] = (b_5 I_3 - b_1 I_1)/d_{15}.$$

(2.36)
$$[1, 1, -1, -1, -2] = (d_{25} I_3 + b_2 I_3')/b_5.$$

(2.37)
$$[1, -1, -1, -3, -2] = (b_5 I_3 - b_4 I_2)/d_{45}.$$

(2.38)
$$[1, 1, -1, -3, -2] = (d_{25} I_3 - d_{24} I_2)/d_{45}.$$

(2.39)
$$[-1, -1, -1, -3, -2] = (b_5^2/d_{15}d_{45})I_3 - (b_4^2/d_{14}d_{45})I_2 \\ + (b_1^2/d_{14}d_{15})I_1.$$

(2.40)
$$[1, 1, 1, -3, -2] = (d_{25}d_{35}/b_5 d_{45})I_3 + (b_2 b_3/b_4 b_5)I_3' \\ - (d_{24}d_{34}/b_4 d_{45})I_2.$$

(2.41)
$$[3, -1, -1, -1, -2] = (d_{15} I_3 + b_1 I_3')/b_5.$$

(2.42)
$$[3, 1, -1, -1, -2] = (d_{15}d_{25}/b_5^2)I_3 + (b_1 b_2/2b_5) \\ \cdot [(r_{13} + 2r_{15} + r_{24})I_3' + b_4 r_{24} r_{34} I_2 - b_1 r_{12} r_{13} I_1] \\ + (b_1/b_3 b_5)A(1, 1, 1, -1).$$

(2.43)
$$[1, 1, 1, -1, -2] = (d_{25}d_{35}/b_5^2)I_3 + (b_2 b_3/2b_5) \\ \cdot [(r_{15} + r_{35} + r_{24})I_3' + b_4 r_{24} r_{34} I_2 - b_1 r_{12} r_{13} I_1] \\ + A(1, 1, 1, -1)/b_5.$$

(2.44)
$$[1, 1, 1, 1, -2] = (d_{25}d_{35}d_{45}/b_5^3)I_3 + (b_2 b_3 b_4/8b_5)(\sigma^2 - 2\tau)I_3' \\ + (\sigma/8b_5)[d_{24}d_{34}I_2 + 2b_4 A(1, 1, 1, -1)] \\ + (b_4 d_{12}d_{13}/8b_1 b_5)(r_{12} - r_{35} - 3r_{45})I_1 \\ + A(1, 1, 1, 1)/2b_5,$$

where

(2.45)
$$\sigma = \sum_{i=1}^{4} r_{i5}, \qquad \tau = \sum_{i=1}^{4} r_{i5}^2.$$

The final five integrals have $p_5 = -4$. All five involve I_3, and the two with $\sum p_i \geq -2$ involve I_3'/b_5^2 also. We assume $a_5 + b_5 t$ is positive on the closed interval of integration.

(2.46)
$$[1, 1, -1, -1, -4] = \frac{1}{2}\left(\frac{d_{23}}{d_{53}} + \frac{d_{24}}{d_{54}} - \frac{d_{21}}{d_{51}}\right)I_3 + \frac{d_{24}d_{34}}{2d_{35}d_{45}}I_2 \\ - \frac{d_{12}d_{13}}{2d_{15}d_{35}}I_1 - \frac{1}{d_{35}}A(1, 1, 1, -1, -2).$$

$$(2.47) \quad [1,-1,-1,-1,-4] = \frac{1}{2}\left(\frac{b_1}{d_{15}} - \sum_{i=2}^{4} \frac{b_i}{d_{i5}}\right) I_3 + \frac{b_5 d_{24} d_{34}}{2 d_{25} d_{35} d_{45}} I_2$$

$$- \frac{b_5 d_{12} d_{13}}{2 d_{15} d_{25} d_{35}} I_1 - \frac{b_5}{d_{25} d_{35}} A(1,1,1,-1,-2).$$

$$(2.48) \quad [1,1,1,-1,-4] = (b_2 b_3/2b_5^2)[b_5(r_{25} + r_{25}r_{35}/r_{15} - r_{24}r_{35}/r_{45}) I_3$$

$$+ 2I_3' + (b_4 r_{24} r_{34}/r_{45}) I_2 - (b_1 r_{12} r_{13}/r_{15}) I_1]$$

$$- A(1,1,1,-1,-2)/b_5.$$

$$(2.49) \quad [-1,-1,-1,-1,-4] = \frac{-1}{2d_{15}} \sum_{i=1}^{4} \frac{1}{r_{i5}} I_3 + \frac{b_5^2 d_{24} d_{34}}{2 d_{15} d_{25} d_{35} d_{45}} I_2$$

$$+ \frac{b_1^2}{d_{15}^2}\left(1 - \frac{r_{12}r_{13}}{2r_{25}r_{35}}\right) I_1$$

$$- \frac{b_5^2}{d_{15} d_{25} d_{35}} A(1,1,1,-1,-2).$$

$$(2.50) \quad [1,1,1,1,-4] = (d_{25} d_{35} d_{45}/2b_5^4) \sum_{i=1}^{4} r_{i5}^{-1} I_3$$

$$+ (b_2 b_3 b_4/2b_5^2) \sum_{i=1}^{4} r_{i5} I_3' + (d_{24} d_{34}/b_5^2) I_2$$

$$- (b_4 d_{12} d_{13}/2b_5 d_{15})(r_{15} + r_{45}) I_1$$

$$- A(1,1,1,1,-2)/b_5 + (2b_4/b_5^2) A(1,1,1,-1).$$

The nine integrals with $p_5 = \pm 2$, $p_6 = \pm 2$, $\sum |p_i| \le 8$, and $\sum p_i \le 0$ can be reduced to those listed above by using the identities

$$(2.51) \quad (a_5 + b_5 t)^{-1}(a_6 + b_6 t)^{-1} = [b_5(a_5 + b_5 t)^{-1} - b_6(a_6 + b_6 t)^{-1}]/d_{65},$$

$$(2.52) \quad (a_5 + b_5 t)^{-1}(a_6 + b_6 t) = [b_6 + d_{65}(a_5 + b_5 t)^{-1}]/b_5,$$

$$(2.53) \quad (a_5 + b_5 t)(a_6 + b_6 t) = [d_{65}(a_5 + b_5 t) + b_6(a_5 + b_5 t)^2]/b_5.$$

3. Quartic Cases of First or Second Kind.

In [4] the nine quartic integrals of first or second kind with $\sum |p_i| \le 8$ were evaluated in terms of R_F and R_D. Since R_D is symmetric in only its first two variables, the third was chosen variously as U_{12}^2, U_{13}^2, or U_{14}^2 to simplify the formulas. In (2.14) and hence throughout Section 2, U_{14}^2 is the choice. For uniformity of notation and for use in deriving the formulas of Section 2, we list again the nine integrals in present notation and add for convenience $[-1,-1,-1,-3]$, which is a special case of $[-1,-1,-1,-3,2]$.

$$(3.1) \quad [-1,-1,-1,-1] = I_1.$$

$$(3.2) \quad [1,-1,-1,-3] = I_2.$$

$$(3.3) \quad [-1,-1,-1,-3] = (b_4 I_2 - b_1 I_1)/d_{14}.$$

$$(3.4) \quad [-1,-1,-1,-3,2] = (d_{54} I_2 + d_{15} I_1)/d_{14}.$$

$$(3.5) \quad [1,1,-3,-3] = [2d_{24} I_2 + d_{12} I_1 + 2A(1,1,-1,-1)]/d_{34}.$$

(3.6)
$$[1, -1, -3, -3] = [(b_3 d_{24} + b_4 d_{23})I_2 + b_3 d_{12} I_1$$
$$+ 2b_3 A(1, 1, -1, -1)]/d_{23} d_{34}.$$

(3.7)
$$[-1, -1, -3, -3] = (1/d_{13})(r_{23}^{-1} - r_{14}^{-1} + 2r_{34}^{-1})I_2$$
$$+ (1/d_{34})(r_{23}^{-1} - r_{14}^{-1})I_1$$
$$+ (2b_3^2/d_{13} d_{23} d_{34})A(1, 1, -1, -1).$$

(3.8)
$$[1, 1, -1, -5] = [(d_{13} d_{24} + d_{14} d_{23})I_2 + d_{12} d_{13} I_1$$
$$+ 2d_{14} A(1, 1, 1, -3)]/3d_{14} d_{43}.$$

(3.9)
$$[1, -1, -1, -5] = (1/3b_4)(r_{14}^{-1} - 2r_{24}^{-1} - 2r_{34}^{-1})I_2$$
$$- (b_4 d_{12} d_{13}/3d_{14} d_{24} d_{34})I_1$$
$$- (2b_4/3d_{24} d_{34})A(1, 1, 1, -3).$$

(3.10)
$$[-1, -1, -1, -5] = (2/3d_{41}) \sum_{i=2}^{4} r_{i4}^{-1} I_2$$
$$+ (b_1/d_{14})^2 (1 - r_{12} r_{13}/3r_{24} r_{34})I_1$$
$$- (2b_4^2/3d_{14} d_{24} d_{34})A(1, 1, 1, -3).$$

4. Reduction by Recurrence Relations.

The integrals in Sections 2 and 3 are expressed in terms of I_1, I_2, I_3, and I_3' by using recurrence relations, most of which were proved in [4]. Let e_i denote an n-tuple with 1 in the ith place and 0's elsewhere (for example, $[p + 2e_1] = [p_1 + 2, p_2, \ldots, p_n]$). We denote [4, (5.4)] by (Ai):

(Ai) $$(p_1 + \cdots + p_n + 2)b_i[p] = \sum_{j \neq i} p_j d_{ji}[p - 2e_j] + 2A(p + 2e_i).$$

For concise reference we write [4, (5.1)] in two forms:

(Bij) $$d_{ij}[p] = b_j[p + 2e_i] - b_i[p + 2e_j],$$

(Cij) $$b_j[p] = b_i[p - 2e_i + 2e_j] + d_{ij}[p - 2e_i].$$

Equation (Cij) is (Bij) with p replaced by $p - 2e_i$. Finally, [4, (5.22)] implies $\sum(a_i + b_i t) d_{jk} = 0$ and hence

(4.1) $$\sum[p + 2e_i] d_{jk} = 0,$$

where $\sum$ denotes summation over cyclic permutations of i, j, k. Replacing p by $p - 2e_k$, we see that

(Dijk) $$d_{ij}[p] = d_{kj}[p + 2e_i - 2e_k] + d_{ik}[p + 2e_j - 2e_k].$$

We discuss first the reduction of the integrals in Section 3, because they will be used to reduce those in Section 2 and because the procedure differs at some points from that used in [4]. Equations (3.1) and (3.2) were proved in [4, §4]. Equation (3.3) obviously comes from (B14) and (3.4) from (D145). For (3.5) we first use (D342) to get

(4.2) $$d_{34}[1, 1, -3, -3] = d_{24} I_2 + d_{32}[1, -1, -3, -1].$$

To reduce the last term, we put $[p] = [1, -1, -1, -1]$ in (A1) and in [4, (5.3)] to find, respectively,

$$(4.3) \quad \begin{aligned} 0 &= d_{12}[1, -3, -1, -1] + d_{13}[1, -1, -3, -1] + d_{14}I_2 \\ &\quad + 2A(3, -1, -1, -1), \end{aligned}$$

$$(4.4) \quad \begin{aligned} b_1 I_1 &- b_2[1, -3, -1, -1] - b_3[1, -1, -3, -1] - b_4 I_2 \\ &= 2A(1, -1, -1, -1). \end{aligned}$$

Elimination of $[1, -3, -1, -1]$ yields

$$(4.5) \quad d_{32}[1, -1, -3, -1] = d_{24}I_2 + d_{12}I_1 + 2A(1, 1, -1, -1),$$

which we have simplified by using (4.8) and [4, (5.22)].

Equation (3.6) follows from (B23) and (3.7) from (B13). Equation (3.8) comes from putting $[p] = [1, 1, -1, -3]$ in (A1); (3.9) then follows from (B24) and (3.10) from (B14).

Equation (2.19) is a special case of (2.34). Equations (2.20) and (2.21) come from (C24) and (C14), respectively. To get (2.22) we use (A3) and evaluate $[-1, 1, -1, -1]$ by (C21). Equations (2.23), (2.24), (2.25), and (2.26) come from (C12), (C34), (C14), and (C34), respectively. For (2.27) we use (A4) and evaluate $[-1, 1, 1, -1]$ by (C31) and $[1, -1, 1, -1]$ by (C32). For (2.28) we use (A1) and evaluate $[1, -1, 1, 1]$ by (C42) and $[1, 1, -1, 1]$ by (C43). Equations (2.29) to (2.33) follow in order from (C51), (C52), (C54), (C54), and (C51).

We shall prove (2.34) in Section 5. Equations (2.35) to (2.44) follow in order from (B15), (C25), (B15), (B45), (D452), (C35), (C15), (C15), (C35), and (C45). To get (2.46) we put $[p] = [1, 1, -1, -1, -2]$ in (A3) and evaluate $[-1, 1, -1, -1, -2]$ by (C21). Equations (2.47) to (2.50) then follow in order from (B25), (C35), (B15), and (C45).

The formulas resulting from this procedure have often been simplified with the help of various identities. Besides [4, (5.22)] we note

$$(4.6) \quad r_{ik} + r_{kj} = r_{ij} = -r_{ji},$$

$$(4.7) \quad r_{ik}^2 + r_{jk}^2 - r_{ij}^2 = 2r_{ik}r_{jk},$$

$$(4.8) \quad d_{ij}A(p) = b_j A(p + 2e_i) - b_i A(p + 2e_j),$$

$$(4.9) \quad \sum A(p + 2e_i)d_{jk} = 0,$$

where $\sum$ denotes summation over cyclic permutations of i, j, k. Equation (4.6) is obvious from (2.1), and (4.7) comes from squaring (4.6). Equation (4.8) follows from (2.6) and [4, (5.7)], and the proof of (4.9) is like that of (4.1).

5. The Fundamental Integral. Equations (2.13) and (2.14), with left sides identified in (2.17), were proved in [4, Section 4], initially with a restriction on the a's and b's that was later removed by using the analyticity of the R-function when each of its variables lies in the complex plane cut along the nonpositive real axis. The proof of (2.15) is similar except for a complication: The variables W^2 and Q^2 are sometimes real and negative.

In this section we assume $x > y$ and $a_i + b_i t > 0$, $y \le t \le x$, for $1 \le i \le 5$. From [4, (4.1), (4.3)] and (1.3) we find the companion of [4, (4.5), (4.6)]:

$$(5.1) \quad I_3 = [1, -1, -1, -1, -2] = (2/3d_{15})(d_{12}d_{13}d_{14})^{-1/2}$$
$$\cdot [R_J(z_2, \ldots, z_5) - R_J(z_2 + \lambda, \ldots, z_5 + \lambda)],$$

where, for $i = 2, \ldots, 5$,

$$(5.2) \quad z_i = Y_i^2/Y_1^2 \, d_{1i}, \quad \lambda = (x - y)/X_1^2 Y_1^2, \quad z_i + \lambda = X_i^2/X_1^2 \, d_{1i}.$$

As in [4], this result depends on the temporary assumption that $-a_1/b_1$ is the first singularity of the integrand to the right of the interval of integration, which implies $d_{1i} > 0$, $i > 1$, and thus guarantees that all variables of R_J are positive. We now apply the addition theorem [9, (8.11)],

$$(5.3) \quad R_J(z_2, \ldots, z_5) = R_J(z_2 + \lambda, \ldots, z_5 + \lambda) + R_J(z_2 + \mu, \ldots, z_5 + \mu)$$
$$+ 3R_C(\gamma + \delta, \gamma),$$

$$(5.4) \quad z_i + \mu = \lambda^{-2}\{[(z_i + \lambda)z_j z_k]^{1/2} + [z_i(z_j + \lambda)(z_k + \lambda)]^{1/2}\}^2,$$

$$(5.5) \quad \gamma = z_5(z_5 + \lambda)(z_5 + \mu), \qquad \delta = (z_2 - z_5)(z_3 - z_5)(z_4 - z_5).$$

In (5.4), i, j, k is any permutation of 2, 3, 4. Equation (5.1) becomes

$$(5.6) \quad I_3 = (2/3d_{15})(d_{12}d_{13}d_{14})^{-1/2}[R_J(z_2 + \mu, \ldots, z_5 + \mu) + 3R_C(\gamma + \delta, \gamma)].$$

From (5.4) and (5.2) we find, for $i = 2, 3, 4$,

$$z_i + \mu = U_{1i}^2/d_{12}d_{13}d_{14},$$
$$(5.7) \qquad z_5 + \mu = (z_i + \mu) + (z_5 - z_i),$$
$$z_5 - z_i = (d_{1i}Y_5^2 - d_{15}Y_i^2)/d_{15}d_{1i}Y_1^2 = -d_{i5}/d_{15}d_{1i},$$

where [4, (5.23)] is used in the last step. It follows that

$$(5.8) \quad z_5 + \mu = W^2/d_{12}d_{13}d_{14},$$

$$(5.9) \quad \gamma = Q^2/d_{12}d_{13}d_{14}d_{15}^2, \qquad \gamma + \delta = P^2/d_{12}d_{13}d_{14}d_{15}^2,$$

where W^2, P^2, and Q^2 are defined by (2.4) and (2.5). Homogeneity of R_J and R_C [see (1.4) and (1.6)] now yields (2.15) from (5.6).

Since z_2, z_3, z_4, λ are positive by (5.2), expansion of the right-hand side of (5.4) shows that $\mu > 0$. Since $z_5 > 0$ by (5.2), it follows by (5.8) that $W^2 > 0$ and by (2.5) that $Q^2 > 0$. Equation (2.8) shows that $P^2 > 0$. Thus the R-functions in (5.6) and (2.15) are well defined when the temporary assumption about $-a_1/b_1$ is satisfied. When it is not, the functions in (2.15) remain well defined if W^2 and Q^2, which have the same signs by (2.5), are positive.

To examine the sign of W^2, we let $w = X_1^2 = a_1 + b_1 x$ and allow w to vary while fixing x and y and $X_i > 0$, $2 \le i \le 5$, and $Y_i > 0$, $1 \le i \le 5$. If we define

$$(5.10) \qquad w_i = X_i^2 Y_1^2/Y_i^2,$$

then $w_2, \ldots, w_5$ are fixed positive quantities, $w = w_1$ is a positive variable, and d_{1i} is a linear function of w because

$$(5.11) \qquad (x - y)d_{1i} = X_i^2 Y_1^2 - Y_i^2 X_1^2 = Y_i^2(w_i - w).$$

We shall want also, for $i = 2, 3, 4$,

(5.12) $$(x - y)d_{i5} = X_5^2 Y_i^2 - Y_5^2 X_i^2 = (Y_i Y_5 / Y_1)^2 (w_5 - w_i).$$

Since Q^2 and W^2 are positive by (2.5) if $d_{25} d_{35} d_{45} = 0$, we may assume that w_5 is not equal to w_2, w_3, or w_4.

The graph of $a_1 + b_1 t$ has the fixed positive ordinate Y_1^2 at $t = y$, the variable positive ordinate w at $t = x$, and the intercept $-a_1/b_1$ on the t-axis. As w runs through all positive real values, this intercept takes all values outside the fixed interval $[y, x]$ of integration. When $w = w_i$ for some $i > 1$, the intercept coincides with the fixed intercept $-a_i/b_i$ of the graph of $a_i + b_i t$. The temporary assumption about $-a_1/b_1$ means that $0 < w < w_i$, $2 \leq i \leq 5$.

We find from (2.4) and (2.3) that

(5.13)
$$W^2 = [Y_2 Y_3 Y_4 / Y_1 (x - y)]^2 f(w),$$
$$f(w) = [(w w_2)^{1/2} + (w_3 w_4)^{1/2}]^2 - (w_3 - w)(w_4 - w)(w_5 - w_2)/(w_5 - w).$$

This is positive when w equals w_2, w_3, or w_4, but it changes sign as w goes through the singularity at w_5. Writing ς for the positive square root $w^{1/2} = X_1$, we find

(5.14)
$$\begin{aligned}(w - w_5)f(w) = {}& w_5 \varsigma^4 + 2(w_2 w_3 w_4)^{1/2} \varsigma^3 \\ & + [w_2 w_3 + w_2 w_4 + w_3 w_4 - w_5(w_2 + w_3 + w_4)] \varsigma^2 \\ & - 2 w_5 (w_2 w_3 w_4)^{1/2} \varsigma - w_2 w_3 w_4.\end{aligned}$$

By Descartes' rule of signs the polynomial in ς has exactly one positive zero, say $\varsigma_0 = w_0^{1/2}$. Thus $f(w)$ changes sign exactly twice, at w_0 and w_5, as w goes through all positive values. Since f is positive at 0 and ∞, it must be negative in the open interval with endpoints w_0 and w_5, either of which may be the greater. This interval cannot contain w_2, w_3, or w_4 since f is positive at those points.

We shall now establish (2.15) for all $w > 0$ by analytic continuation. We cut the w-plane along the nonpositive real axis and make a second cut between w_0 and w_5 on the positive real axis. The corresponding region of the ς-plane is the open right half-plane with a cut between $w_0^{1/2}$ and $w_5^{1/2}$ on the positive real axis. The coefficient of R_J in (2.15) and the variables of both R_J and R_C are analytic functions of w in the cut w-plane by (2.3), (2.4), (2.5), (5.11), and (5.12). We recall that an R-function is analytic when each of its variables lies in the complex plane cut along the nonpositive real axis [2, (6.8-6), Theorem 6.8-1]. Therefore, if we can show that none of the variables can be real and nonpositive when w is in the cut w-plane, it will follow that R_J, R_C, and the right-hand side of (2.15) are analytic.

Since U_{12}, U_{13}, and U_{14} are linear functions of ς with positive coefficients by (2.3), they have positive real parts when ς does, and their squares cannot be real and nonpositive. A similar remark, with ς replaced by $1/\varsigma$, applies to P^2 by (2.8). It remains to consider W^2 and Q^2.

Now W^2 is real and nonpositive if and only if $f(w) + r = 0$ for some $r \geq 0$. The quantity

(5.15) $$(w - w_5)[f(w) + r] = (w - w_5)f(w) + r\varsigma^2 - r w_5$$

is a quartic polynomial in ς whose coefficients, like those of (5.14), have the signs

(5.16) $$+, \ +, \ \pm, \ -, \ -.$$

By Descartes' rule of signs there is exactly one positive zero, which must lie on the cut between $w_0^{1/2}$ and $w_5^{1/2}$ where $f(w)$ takes every nonpositive real value. Either the other three zeros are real and negative or else one is negative and the other two are conjugate complex. In the latter case the polynomial must have the form

$$(5.17) \qquad \begin{aligned} w_5(\varsigma - \xi)(\varsigma &+ \eta)(\varsigma - \alpha - i\beta)(\varsigma - \alpha + i\beta) \\ &= w_5[\varsigma^2 + (\eta - \xi)\varsigma - \xi\eta][\varsigma^2 - 2\alpha\varsigma + \alpha^2 + \beta^2], \end{aligned}$$

where ξ and η are positive and α and β are real. The coefficients of ς^3 and ς have the signs required by (5.16) if and only if

$$(5.18) \qquad \eta - \xi - 2\alpha > 0, \qquad (\alpha^2 + \beta^2)(\eta - \xi) + 2\alpha\xi\eta < 0,$$

which imply $2\alpha(\alpha^2 + \beta^2 + \xi\eta) < 0$ and hence $\alpha < 0$. Thus the conjugate complex roots have negative real part. We conclude that a quartic polynomial whose coefficients have the signs (5.16) has one real positive zero and three zeros in the open left half-plane. Thus no zero of (5.15) lies in the cut right-half ς-plane, and W^2 cannot be real and nonpositive at any point in the cut w-plane.

By (2.5) and (5.13), Q^2 is a positive multiple of $f(w)/w$. Since $w \neq 0$ in the cut plane, Q^2 is real and nonpositive if and only if $f(w) + rw = 0$ for some $r \geq 0$. The quantity

$$(5.19) \qquad (w - w_5)[f(w) + rw] = (w - w_5)f(w) + r\varsigma^4 - w_5 r\varsigma^2$$

is a quartic polynomial in ς whose coefficients, like those of (5.14), have the signs (5.16). Thus Q^2, like W^2, cannot be real and nonpositive at any point of the cut plane. This completes the proof that the right-hand side of (2.15) is analytic in w on the cut w-plane.

The left side of (2.15) is defined by (1.3), which can be rewritten in the form of [4, (4.12)]:

$$(5.20) \qquad \begin{aligned} I_3 = [1, -1, -1, -1, -2] &= (x - y)(Y_1/Y_2 Y_3 Y_4 Y_5^2) \\ &\cdot R_{-1}\left(\frac{-1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1; \frac{w}{Y_1^2}, \frac{X_2^2}{Y_2^2}, \dots, \frac{X_5^2}{Y_5^2}\right). \end{aligned}$$

This is analytic in the w-plane cut along the nonpositive real axis, even without the second cut from w_0 to w_5. Since (2.15) is known to hold if $0 < w < w_i$, $2 \leq i \leq 5$, it holds throughout the twice-cut w-plane by the permanence of functional relations. Each of the two terms on the right-hand side of (2.15) is discontinuous across the second cut, but the discontinuities must cancel (as can be verified explicitly) because the left side is continuous. Hence the second cut is unnecessary and can be removed after defining R_J and R_C on the cut so as to make the right-hand side continuous. A suitable definition for each is its Cauchy principal value, which is the arithmetic average of its values on the upper and lower edges of the cut. With this definition, (2.15) holds on the w-plane cut only along the nonpositive real axis. In particular it holds for all positive values of w, which was to be proved.

At each endpoint of the second cut the Cauchy principal value is to be interpreted as a limit of values off the cut or of principal values on the cut. The limit is easy to evaluate at the end where $w = w_5$, since this implies by (5.11) that d_{15} vanishes, $a_5 + b_5 t$ is proportional to $a_1 + b_1 t$, and $[1, -1, -1, -1, -2]$ reduces to $(a_1/a_5)[-1, -1, -1, -1]$.

6. Cauchy Principal Values. According to Section 5, the evaluation of I_3 may involve Cauchy principal values of both R_J and R_C, even though $a_5 + b_5 t$ is positive on the closed interval of integration. We shall prove now that formulas (2.34) to (2.44) are still valid if $a_5 + b_5 t$ changes sign in the open interval of integration. If the left side of each formula is taken to be a Cauchy principal value, then I_3 as defined by (2.15) involves the Cauchy principal value of either R_J or R_C but not both, since W^2 and Q^2 have opposite signs by (2.5) when $X_5^2 Y_5^2 < 0$.

It suffices to consider (2.34), since the proofs of (2.35) to (2.44) by recurrence relations (Bij), (Cij), and ($Dijk$) do not depend on the sign of $a_5 + b_5 t$. We shall assume $a_5 + b_5 x < 0 < a_5 + b_5 y$, but the proof with x and y interchanged is similar. The method is much like that in Section 5.

We fix x and y and $X_i > 0$, $1 \leq i \leq 4$ and $Y_i > 0$, $1 \leq i \leq 5$, while allowing $X_5^2 = a_5 + b_5 x$ to vary in the complex plane. By (5.10), $w_1, \ldots, w_4$ are fixed positive quantities while $w_5 = X_5^2 Y_1^2 / Y_5^2$ is variable. Since w is an abbreviation for w_1, we see from (5.11) that the coefficient of R_J in (2.15) is analytic except for a pole at w_1. The first three arguments of R_J are fixed and positive, but it follows from (5.13) that the fourth argument has the form

$$(6.1) \qquad\qquad W^2 = A + \frac{B}{w_5 - w_1},$$

where A and B are real and independent of w_5. A little algebra shows that $A > 0$ if $B = 0$, and hence W^2 cannot be real and nonpositive unless v''_5 is real. We cut the w_5-plane along the nonpositive real axis and also along every interval of the positive real axis where $W^2 \leq 0$. Unless $B = 0$, one of these intervals will have the pole at w_1 as one endpoint, and so the cut plane will in all cases include an open interval of the positive real axis, where $X_5^2 > 0$ and (2.15) is valid by Section 5. In the cut plane, W^2 is analytic in w_5 and cannot be real and nonpositive. Hence, the first term on the right-hand side of (2.15) is analytic in w_5 on the cut plane.

We rewrite the second term, using the homogeneity of R_C, as

$$(6.2) \qquad\qquad 2w_5^{-1/2} R_C((Pw_5^{-1/2})^2, Q^2/w_5).$$

Equation (2.8) shows that

$$(6.3) \qquad\qquad Pw_5^{-1/2} = Mw_5^{-1/2} + Nw_5^{1/2},$$

where M and N are positive and independent of w_5. If w_5 is in the cut plane, both terms on the right-hand side of (6.3) lie in the open right half-plane, and hence $(Pw_5^{-1/2})^2$ cannot be real and nonpositive. Neither can Q^2/w_5, since it is a positive multiple of W^2 by (2.5). Thus, the second term on the right-hand side of (2.15) also is analytic in w_5 on the cut plane.

Finally, the left side of (2.15) is analytic in w_5 on the cut plane because the first four arguments of R_{-1} in (5.20) are fixed and positive while the last argument is a positive multiple of w_5. Since the cut plane contains an open interval of the positive real axis and (2.15) holds on that interval, it holds everywhere in the cut plane by the permanence of functional relations. The equality clearly persists if each term in the equation is replaced by the arithmetic average of its values on the upper and lower edges of the cut along the negative real axis. On the left side this average is the Cauchy principal value of I_3. On the right-hand side, either R_J or R_C takes

its Cauchy principal value, but since Q^2 and W^2 have opposite signs when w_5 is negative, one of the two functions will be continuous across the cut.

A numerical example in which I_3 and R_C take their principal values is given at the end of the Supplement.

Ames Laboratory and Department of Mathematics
Iowa State University
Ames, Iowa 50011

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